Date-27-04-2020, Remaining Part —

Topic — Differentiability and Differential

State and Prove SCHWARTZ'S THEOREM

Statement :→ Let (a,b) be a point of the domain of a function f such that

(i) f_x exists in certain nhd. of (a,b)

(ii) f_{xy} is continuous at (a,b)

then $f_{yx}(a,b)$ exists and

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Proof :→ The given condition imply that there exists a certain nhd. of (a,b) at every point (x,y) of which $f_x(x,y)$, $f_y(x,y)$ and $f_{xy}(x,y)$ all exists

Let $(a+h, b+k)$ be any point of the nhd.

We write $F(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$

$$g(y) = f(a+h, y) - f(a,y)$$

So that $F(h,k) = g(b+k) - g(b)$ ——— (1)

Since f_y exists in nhd of (a,b), the function g of one variable y is derivable in $[b, b+k]$ and therefore, By applying the Mean-value theorem to the Expansion on the right of (1).

we have

$F(h,k) = k g'(b+\theta k)$ where $0<\theta<1$

$$= k\left[f_y(a+h, b+\theta k) - f_y(a, b+\theta k)\right] \quad \text{——— (2)}$$

Again since f_{xy} Exists in a nhd. of (a,b), therefore by applying the Mean value theorem to the right of (2)

$F(h,k) = k\left[h f_{xy}(a+\theta' h, b+\theta k)\right]$ where $0<\theta'<1$

$\Rightarrow \frac{F(h,k)}{hk} = f_{xy}(a+\theta' h, b+\theta k)$

$\Rightarrow \frac{1}{k}\left[\frac{f(a+h,b+k)-f(a,b+k)}{h} - \frac{f(a+h,b)-f(a,b)}{h}\right]$

$= f_{xy}(a+\theta' h, b+\theta k)$

Since f_x exists in a nbd. of (a,b), Therefore on taking the limit as $h \to 0$, we get

$\frac{1}{k}\left[f_x(a,b+k) - f_x(a,b)\right] = \lim_{h\to 0} f_{xy}(a+\theta' h, b+\theta k)$

Now let $k \to 0$

Then $\lim_{k\to 0} \frac{f_x(a,b+k)-f_x(a,b)}{k} = \lim_{h\to 0} f_{xy}(a+\theta' h, b+\theta k)$

$\Rightarrow \cancel{f_{yx}(a,b)}\ f_{yx}(a,b) = f_{xy}(a,b)$

Since f_{xy} is continuous at (a,b) (Proved)

Cor. $\rightarrow$ If f_{xy} and f_{yx} both are continuous at (a,b)

Then $f_{xy}(a,b) = f_{yx}(a,b)$

State and Prove young's theorem →

Statement :→ Let (a,b) be a point of the domain of a function f such that f_x and f_y both are differentiable at (a,b).

then $f_{xy}(a,b) = f_{yx}(a,b)$

Proof :→ The differentiability of f_x and f_y implies that they exist in a certain nhd of (a,b) such that

$$f_{xx},\ f_{yx},\ f_{xy},\ f_{yy} \text{ exist at } (a,b)$$

we write

$$F(h,h) = f(a+h,b+h) - f(a+h,b) - f(a,b+h) + f(a,b)$$

$$g(y) = f(a+h,y) - f(a,y)$$

so that

$$F(h,h) = g(b+h) - g(b) \quad \text{——— (1)}$$

Since f_y exists in a nhd of (a,b) therefore, By applying the Mean-value theorem to the expression on the R.H.S of (1), we obtain

$$F(h,h) = h\,g'(b+\theta h) \quad \text{where } 0<\theta<1$$

$$= h\left[f_y(a+h,b+\theta h) - f_y(a,b+\theta h)\right] \quad \text{——— (2)}$$

Since f_y is differentiable at (a,b), therefore, By definition we have

$$f_y(a+h,b+\theta h) - f_y(a,b) = h f_{xy}(a,b) + \theta h\cdot f_{yy}(a,b) + h\varphi_1(h,h) + \theta h\psi_1(h,h) \quad \text{——— (3)}$$

Where

$\varphi_1(h,h)$ and $\psi_1(h,h) \longrightarrow 0$ as $h \to 0$.

Similarly $f_y(a, b+\theta h) - f_y(a,b) = \theta h f_{yy}(a,b) + \theta h \psi_2(h,h)$ — (4)

Where $\psi_2(h,h) \longrightarrow 0$ as $h \to 0$

Now (3)-(4) gives.

$$f_y(a+h, b+\theta h) - f_y(a, b+\theta h) = h f_{xy}(a,b) + h\varphi_1(h,h) + \theta h\{\psi_1(h,h) - \psi_2(h,h)\}$$

From (2) we have

$$\Rightarrow \frac{F(h,h)}{h} = \quad , , \quad , , \quad , , \quad \text{from (2)}$$

$$\Rightarrow \frac{F(h,h)}{h^2} = f_{xy}(a,b) + \varphi_1(h,h) + \theta\{\psi_1(h,h) - \psi_2(h,h)\} \text{ — (5)}$$

By a similar argument and considering

$u(x) = f(x, b+h) - f(x,b)$

we may show that

$$\frac{F(h,h)}{h^2} = f_{yx}(a,b) + \theta'\{\varphi_2'(h,h) - \psi_2'(h,h) + \psi_1'(h,h)\}$$

Where $\varphi_2', \psi_2', \psi_1' \longrightarrow 0$ as $h \to 0$. — (6)

Equating the R.H.S of (5) and (6)

and making $h \to 0$, we get.

$$f_{xy}(a,b) = f_{yx}(a,b)$$